

Constraints on Neutrino Mixing

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Abstract

We explore the implications of imposing the constraint that two neutrino flavors (which for definiteness we take to be ν_μ and ν_τ) are similarly coupled to the mass basis in addition to the unitarity constraints. We allow three active and an arbitrary number of sterile neutrinos. We show that in this scheme one of the mass eigenstates decouples from the problem, reducing the dimension of the flavor space by one.

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I. INTRODUCTION

Recent observations of atmospheric neutrinos at the SuperKamiokande experiment [1] very strongly suggest that the muon neutrino maximally mixes with another neutrino, which is either the tau neutrino or a sterile neutrino. In a parallel development Elwood, Irges, and Ramond suggested that the observed quark hierarchies indicate a simple family symmetry [2]. They argue that the manifest interfamily hierarchy, for example in the Wolfenstein parameterization [3] of the CKM matrix, points to the existence of an anomalous $U(1)$ family symmetry beyond the Standard Model. They conclude that the MNS neutrino mixing matrix [4] will be of the form

$$\begin{pmatrix} 1 & \lambda^3 & \lambda^3 \\ \lambda^3 & 1 & 1 \\ \lambda^3 & 1 & 1 \end{pmatrix}, \quad (1.1)$$

where λ is the Cabibbo angle. Motivated by this work and the Superkamiokande results in this paper we wish to explore the implications of the constraint that two neutrino flavors (which for illustrative purposes we take to be ν_μ and ν_τ) are similarly coupled to the mass basis.

The possibility of the existence of light sterile neutrinos was suggested not only by the atmospheric neutrino measurements, but by the LSND experiment [5] and by the recent work [6,7] on the possibility of an active-sterile neutrino transformation enabling the production of the r -process nuclei in neutrino-heated supernova ejecta [8]. Hence we consider three active flavors and an arbitrary number (which could be taken to be zero) of sterile neutrinos. At this point we do not specify neutrino mass eigenvalues and allow all possible schemes. The $N \times N$ neutrino mixing matrix will be denoted by $U_{\alpha i}$ where α denotes the flavor index and i denotes the mass index:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle. \quad (1.2)$$

Note that even though we want to impose $|U_{\mu i}| \sim |U_{\tau i}|$ we cannot simply take $U_{\mu i} = U_{\tau i}$ as that gives $\det U = 0$, in contradiction with unitarity which requires $\det U = 1$. Instead we impose the next simplest constraint, namely that they are proportional for all but one mass eigenstate, which we choose for definiteness to be the third mass eigenstate:

$$U_{\mu i} \sim U_{\tau i} \neq 0, \forall i \neq 3. \quad (1.3)$$

We write this condition in terms of an arbitrary angle ϕ and an arbitrary phase η :

$$\sin \phi U_{\mu i} = e^{i\eta} \cos \phi U_{\tau i} \neq 0, \forall i \neq 3. \quad (1.4)$$

Note that, in our formalism, we permit CP-violating phases. Introducing the quantity

$$A = \sum_{i \neq 3} [|U_{\mu i}|^2 + |U_{\tau i}|^2] \quad (1.5)$$

and using Eq. (1.4) one can easily show that

$$\sum_{i \neq 3} |U_{\mu i}|^2 = A \cos^2 \phi, \quad (1.6)$$

and

$$\sum_{i \neq 3} |U_{\tau i}|^2 = A \sin^2 \phi. \quad (1.7)$$

Eqs. (1.6) and (1.7) along with the unitarity constraints $\sum_i |U_{\mu i}|^2 = 1 = \sum_i |U_{\tau i}|^2$ immediately yield

$$|U_{\mu 3}|^2 = 1 - A \cos^2 \phi, \quad (1.8)$$

and

$$|U_{\tau 3}|^2 = 1 - A \sin^2 \phi. \quad (1.9)$$

Inserting Eqs. (1.8) and (1.9) into the unitarity constraint $\sum_i U_{\mu i} U_{\tau i}^* = 0$ and using the constraint $\sum_i |U_{\mu i}|^2 = 1$ we obtain

$$A = 1, \quad (1.10)$$

$$U_{\mu 3} = -\sin \phi e^{i\delta} e^{i\eta}, \quad (1.11)$$

and

$$U_{\tau 3} = \cos \phi e^{i\delta}, \quad (1.12)$$

where δ is a phase to be determined. Furthermore one can write down the unitarity constraints $\sum_i U_{\mu i} U_{\alpha i}^* = 0$ and $\sum_i U_{\tau i} U_{\alpha i}^* = 0$, where α stands for either the electron neutrino or one of the sterile neutrinos. Inserting Eqs. (1.4), (1.11), and (1.12) into these constraints and subtracting them gives

$$U_{\alpha 3} = 0, \quad \alpha \neq \mu, \tau, \quad (1.13)$$

which implies

$$|\nu_\alpha\rangle = \sum_{i \neq 3} U_{\alpha i} |\nu_i\rangle, \quad \alpha \neq \mu, \tau. \quad (1.14)$$

This is a remarkable result. Simply imposing the restriction in Eq. (1.3) and taking into account the unitarity of the neutrino mixing matrix decouples all the other flavors from the third mass eigenstate. Similarly introducing the states

$$|\tilde{\nu}_\mu\rangle = \cos \phi |\nu_\mu\rangle + \sin \phi e^{i\eta} |\nu_\tau\rangle, \quad (1.15)$$

and

$$|\tilde{\nu}_\tau\rangle = -\sin \phi e^{-i\eta} |\nu_\mu\rangle + \cos \phi |\nu_\tau\rangle, \quad (1.16)$$

one can easily show that

$$|\tilde{\nu}_\mu\rangle = \frac{1}{\cos\phi} \sum_{i \neq 3} U_{\mu i} |\nu_i\rangle, \quad (1.17)$$

and

$$|\tilde{\nu}_\tau\rangle = e^{i\delta} |\nu_3\rangle. \quad (1.18)$$

The combination of flavor states in Eq. (1.16) is a pure mass eigenstate and is decoupled from all the physical processes except the $\nu_\mu \rightleftharpoons \nu_\tau$ oscillations (which could be taken to fit the atmospheric neutrino results). The mixing of the remaining neutrinos is governed by Eqs. (1.14) and (1.17), i.e. the original N flavor mixing problem is reduced to a problem where $N - 1$ flavors (the electron neutrino, $\tilde{\nu}_\mu$, and the sterile neutrinos) are mixed.

II. MIXING OF THREE ACTIVE FLAVORS

If we have only three active flavors of neutrinos mixing with no contribution from the sterile neutrinos the state $|\tilde{\nu}_\tau\rangle$ decouples and the two states

$$\begin{pmatrix} |\nu_e\rangle \\ |\tilde{\nu}_\mu\rangle \end{pmatrix} \quad (2.1)$$

result from transforming the mass eigenstates with the matrix

$$\begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1}/\cos\phi & U_{\mu 2}/\cos\phi \end{pmatrix}. \quad (2.2)$$

The solar neutrino data in this case could be explained by either the matter-enhanced or vacuum $\nu_e \rightarrow \tilde{\nu}_\mu$ oscillation. Since both ν_μ and ν_τ gain the same effective mass due to coherent forward scattering in matter, standard MSW analyses for $\nu_e \rightarrow \nu_\mu$ conversion [9] is also valid for $\nu_e \rightarrow \tilde{\nu}_\mu$ conversion. For the solar neutrino experiments where mu and tau neutrinos are detected (such as SuperKamiokande and SNO) it is not possible to distinguish ν_μ and $\tilde{\nu}_\mu$ as at the low solar energies both of these neutrinos have the same neutral current interactions with electrons and deuterons. The recoil electron kinetic energy spectra for ν_e and $\tilde{\nu}_\mu$ are however different. This difference can be used to determine the solar $\tilde{\nu}_\mu$ component at BOREXINO or KamLAND [10].

In the special case $\phi = \pi/4$ the mixing matrix for the three active flavors is

$$\begin{pmatrix} U_{e1} & U_{e2} & 0 \\ \sqrt{2}U_{\mu 1} & \sqrt{2}U_{\mu 2} & \frac{1}{\sqrt{2}}e^{i\delta} \\ \sqrt{2}U_{\mu 1} & \sqrt{2}U_{\mu 2} & -\frac{1}{\sqrt{2}}e^{i\delta} \end{pmatrix}. \quad (2.3)$$

Unitarity constraints imply that the remaining matrix elements can be written in terms of a single mixing angle:

$$\begin{aligned} U_{e1} &= \cos\theta, & U_{e2} &= -\sin\theta, \\ U_{\mu 1} &= \sin\theta, & U_{\mu 2} &= \cos\theta. \end{aligned} \quad (2.4)$$

This is the neutrino mixing matrix given in Ref. [11]. Setting $\theta = \pi/4$ and $\delta = 0$ yields bi-maximal mixing of three active neutrinos [12,13]. With a suitably-chosen mass hierarchy bi-maximal neutrino mixing matrix can successfully explain observations of the atmospheric muon neutrino deficit and can rule out the small-angle MSW solution of the solar neutrino data if relic neutrinos contribute more than one percent to the closure density of the universe [14].

III. MIXING OF AN ARBITRARY NUMBER OF FLAVORS

In this case the states

$$\begin{pmatrix} |\nu_e\rangle \\ |\tilde{\nu}_\mu\rangle \\ |\nu_s\rangle \\ . \end{pmatrix} \quad (3.1)$$

are obtained by transforming the mass eigenstates with the matrix

$$T = \begin{pmatrix} U_{e1} & U_{e2} & U_{e4} & . \\ U_{\mu 1}/\cos\phi & U_{\mu 2}/\cos\phi & U_{\mu 4}/\cos\phi & . \\ U_{s1} & U_{s2} & U_{s4} & . \\ . & . & . & . \end{pmatrix}. \quad (3.2)$$

The constraint we have adopted, Eq. (1.4), is motivated by recent experimental results. It is possible to put this in a form which may motivate model building. In general N flavors of neutrinos mix under the fundamental representation of $U(N)$. A general element of $U(N)$ can be written as a product of $N(N-1)/2$ different, non-commuting $SU(2)$ rotations and a diagonal matrix with matrix elements which are pure phases. (In the familiar $U(3)$ case these $SU(2)$ groups are known as i-, u-, and v-spin [15]). Thus, for N flavors, one can write

$$U_{i\alpha}^\dagger = \begin{pmatrix} C_{12} & S_{12}^* & 0 & . \\ -S_{12} & C_{12}^* & 0 & . \\ 0 & 0 & 1 & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13}^* & . \\ 0 & 1 & 0 & . \\ -S_{13} & 0 & C_{13}^* & . \\ . & . & . & . \end{pmatrix} \cdots \begin{pmatrix} e^{i\delta_1} & 0 & 0 & . \\ 0 & e^{i\delta_2} & 0 & . \\ 0 & 0 & e^{i\delta_3} & . \\ . & . & . & . \end{pmatrix}. \quad (3.3)$$

Here we label each $SU(2)$ rotation R_{ab} , $a < b$, $a, b = 1, \dots, N$. The aa -th, bb -th, ab -th, and ba -th elements of R_{ab} are C_{ab} , C_{ab}^* , $-S_{ab}^*$, and S_{ab} , respectively. In all these matrices the condition $|C_{ab}|^2 + |S_{ab}|^2 = 1$ is satisfied. The phases $\delta_1, \delta_2, \dots$ may be absorbed into the mass eigenstates. Our constraint is equivalent to choosing $C_{a3} = 1$ for all a except for $a = 2$. For this case we have $C_{23} = \cos\phi$ and $S_{23} = e^{i\eta} \sin\phi$. This reduces the original $N(N-1)/2$ parameters into $(N^2 - 3N + 4)/2$ parameters.

Either using Eq. (3.3) or the conditions derived in the Introduction one can easily show that the matrix T of Eq. (3.2) is also unitary. Using this result one can write down the equation that governs the evolution of flavor states in matter [16]. For the special case of two active (ν_e and $\tilde{\nu}_\mu$) and one sterile (ν_s) flavor states (cf. Eq. (3.1)) one gets

$$\begin{aligned}
i \frac{\partial}{\partial x} \begin{pmatrix} |\nu_e\rangle \\ |\tilde{\nu}_\mu\rangle \\ |\nu_s\rangle \end{pmatrix} = & \\
\begin{pmatrix} \Delta_2 |U_{e2}|^2 + \Delta_4 |U_{e4}|^2 + V_e & \frac{1}{\cos \phi} [\Delta_2 U_{e2} U_{\mu 2}^* + \Delta_4 U_{e4} U_{\mu 4}^*] & \Delta_2 U_{e2} U_{s2}^* + \Delta_4 U_{e4} U_{s4}^* \\ \frac{1}{\cos \phi} [\Delta_2 U_{e2}^* U_{\mu 2} + \Delta_4 U_{e4}^* U_{\mu 4}] & \frac{1}{\cos^2 \phi} [\Delta_2 |U_{\mu 2}|^2 + \Delta_4 |U_{\mu 4}|^2] + V_\mu & \frac{1}{\cos \phi} [\Delta_2 U_{\mu 2} U_{s2}^* + \Delta_4 U_{\mu 4} U_{s4}^*] \\ \Delta_2 U_{e2}^* U_{s2} + \Delta_4 U_{e4}^* U_{s4} & \frac{1}{\cos \phi} [\Delta_2 U_{\mu 2}^* U_{s2} + \Delta_4 U_{\mu 4}^* U_{s4}] & \Delta_2 |U_{s2}|^2 + \Delta_4 |U_{s4}|^2 \end{pmatrix} \\
\times \begin{pmatrix} |\nu_e\rangle \\ |\tilde{\nu}_\mu\rangle \\ |\nu_s\rangle \end{pmatrix}, & \tag{3.4}
\end{aligned}$$

where

$$V_e(x) = \sqrt{2} G_F [N_e(x) - N_n(x)/2], \tag{3.5}$$

$$V_\mu(x) = -\frac{1}{\sqrt{2}} G_F N_n(x), \tag{3.6}$$

and

$$\Delta_i = (m_i^2 - m_1^2) / 2E. \tag{3.7}$$

(Note that we defined Δ_1 to be zero). In the next section we comment on a particular application of this evolution equation.

IV. APPLICATIONS AND CONCLUSIONS

The utility of our constraints and their role in calculating matter enhancement effects is best illustrated with an example from astrophysics. Ref. [7] presents a 4×4 neutrino mass and mixing scheme to solve the neutron deficit problem associated with r -process nucleosynthesis from neutrino-heated supernova ejecta. This neutrino mass scheme has a maximally mixed, or near maximally mixed doublet of ν_μ and ν_τ neutrinos split from a lower mass doublet consisting of ν_e and a sterile species, ν_s . In this model, neutrinos emitted from the surface of a neutron star propagate through two resonances in sequence: (1) a $\nu_{\mu,\tau} \rightleftharpoons \nu_s$ resonance; and (2) a $\nu_{\mu,\tau} \rightleftharpoons \nu_e$ resonance.

Because of matter effects, and because of the nature of the density gradient in supernovae [7], we can approximate each resonance as a 3×3 mixing problem. At the first resonance we envision that the ν_μ and ν_τ are coupled to the mass basis similarly, and each of these states is coupled with the ν_s by a mixing matrix of the form in Eq. (2.3). Likewise, the neutrino states at the second resonance are similarly coupled (up to phases) in the 3×3 sector, but now with the ν_e substituted for the ν_s .

The relevant 3×3 sector of the overall mixing matrix at each resonance can be put in the form of Eq. (2.3) in the following manner. We take $|\nu_3\rangle$ and $|\nu_4\rangle$ as the mass eigenstates to which ν_τ and ν_μ are similarly coupled in each case. The other flavor state, $|\nu_s\rangle$ or $|\nu_e\rangle$, is in addition coupled to mass eigenstate $|\nu_i\rangle$, with $i = 1$ or $i = 2$, respectively. At each of the resonances we can write,

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} \cos \phi & 0 & e^{i\delta} \sin \phi \\ 0 & 1 & 0 \\ e^{-i\delta} \sin \phi & 0 & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.1)$$

In the model of Ref. [7] $\psi = \pi/4$, implying maximal mixing of ν_μ and ν_τ , and $\phi = 0$, to remove CP violation effects. The angle ω can be chosen to ensure adiabatic flavor amplitude evolution in the supernova environment (or to fit the LSND results). We note that this 3×3 mixing is the same as that employed in Ref. [14], but with the opposite order for the rotations. A mixing matrix of this form implies that at each resonance we can write:

$$|\nu_\alpha\rangle = \cos \omega |\nu_i\rangle + \sin \omega |\nu_3\rangle \quad (4.2)$$

$$|\nu_\beta\rangle = \frac{1}{\sqrt{2}} \{-\sin \omega |\nu_i\rangle + \cos \omega |\nu_3\rangle + |\nu_4\rangle\} \quad (4.3)$$

$$|\nu_\gamma\rangle = \frac{1}{\sqrt{2}} \{\sin \omega |\nu_i\rangle - \cos \omega |\nu_3\rangle + |\nu_4\rangle\} \quad (4.4)$$

The relevant 3×3 sector at the first resonance, $\nu_{\mu,\tau} \rightleftharpoons \nu_s$, has $i = 1$, $\alpha = s$, $\beta = \mu$, and $\gamma = \tau$. The second resonance, $\nu_{\mu,\tau} \rightleftharpoons \nu_e$, has $i = 2$, $\alpha = e$, $\beta = \mu$, and $\gamma = \tau$.

Our constraints can now be applied as above to isolate the $|\nu_4\rangle$ mass eigenstate:

$$|\tilde{\nu}_\mu\rangle \equiv \frac{1}{\sqrt{2}} \{|\nu_\mu\rangle - |\nu_\tau\rangle\} = -\sin \omega |\nu_i\rangle + \cos \omega |\nu_3\rangle \quad (4.5)$$

$$|\tilde{\nu}_\tau\rangle \equiv \frac{1}{\sqrt{2}} \{|\nu_\mu\rangle + |\nu_\tau\rangle\} = |\nu_4\rangle \quad (4.6)$$

Let us take the first resonance as an example of how the Mikheyev-Smirnov-Wolfenstein (MSW) [16] mechanism works in the “similar coupling” case. Consider a ν_μ neutrino emitted from the neutron star surface and propagating outward. Assuming the vacuum mixing angle ω is small, then in vacuum $|\tilde{\nu}_\mu\rangle$ will be mostly $|\nu_3\rangle$. However, at high density matter effects could provide a negative contribution to the effective mass of this state, and so cause it to be almost entirely the lighter mass eigenstate $|\nu_1\rangle$. Note that the other component $|\tilde{\nu}_\tau\rangle = |\nu_4\rangle$ is always a mass eigenstate.

If the neutrino flavor amplitudes evolve adiabatically, then a neutrino state which begins as a mass eigenstate will remain in this mass eigenstate, even if neutrinos propagate through mass level crossings. Therefore, in our example, the part of the initial $|\nu_\mu\rangle$ which is $|\tilde{\nu}_\tau\rangle$ propagates directly through the resonance unaltered; whereas, that part of the initial state which is $|\tilde{\nu}_\mu\rangle$ will propagate through the first resonance as a ν_1 , which will be mostly ν_s by the time the neutrino reaches the outside of the star. Therefore, MSW resonances behave like “filters,” passing the $|\tilde{\nu}_\tau\rangle$ state, but converting the $|\tilde{\nu}_\mu\rangle$ state to a flavor different than the initial flavor. This behavior will also hold in the general case where “similar coupling” allows us to isolate a mass eigenstate.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. National Science Foundation Grants No. PHY-9605140 at the University of Wisconsin, and PHY-9800980 at the University of California, San Diego and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation. We thank the Aspen Center for Physics for its hospitality during the early stages of this work. We thank Institute for Nuclear Theory at the University of Washington for their hospitality and Department of Energy for partial support during the completion of this work. We also would like to thank D. O. Caldwell, G. C. McLaughlin, Y.-Z. Qian, and P. Ramond for useful discussions.

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